Primer on Inventory Management – Solutions

Periodic Review Model

Solution to Exercise 1:

	Period					
	1	2	3	4	5	6
1. Determine inventory level I_t	17	11	7	-2	-9	-13
2. Determine open orders O_t	0	0	6	10	19	20
3. Compute inventory position IP_t	17	11	13	8	10	7
4. Compute order quantity x_t	0	6	4	9	7	10
5. Receive shipment x_{t-LT}	0	0	0	0	6	4
6. Fill demand y_t	6	4	9	7	10	5

Solution to Exercise 2:

i. The critical ratio is

$$\frac{p}{p+h} = \frac{10}{10+1} \approx 0.909.$$

Demand over LT+R is Poisson distributed with mean $\mu_{LT+R} = 42 \cdot 9.8 = 411.6$. The distribution function of a poisson distribution with rate μ is

$$p_y = \frac{\mu^y}{y!} \cdot e^{-\mu}$$

The correspondung values should be computed in Excel. Below we give an excerpt(!) from the corresponding spreadsheet:

y	p_y^{LT+R}	P_y^{LT+R}
420	0.0179	0.6720
425	0.0156	0.7548
430	0.0128	0.8245
435	0.0100	0.8800
438	0.0083	0.9066
439	0.0078	0.9144
440	0.0073	0.9217
445	0.00051	0.9512

The optimal order quantity is given by the smallest y for wich the cumulative distribution function of the demand is greater or equal to the critical ratio. In our case, this yields $S^\star = 439$.

The expected cost per review period are given by:

$$Z(S) = h \cdot E[S - Y_{LT+R}]^+ + p \cdot E[Y_{LT+R} - S]^+$$

Expected overage quantity:

$$\begin{aligned} E[S - Y_{LT+R}]^+ &= \sum_{y=0}^{S} (S - y) \cdot p_y^{LT+R} \\ &= 28.264 \end{aligned}$$

Expected underage quantity:

$$\begin{aligned} E[Y_{LT+R} - S]^+ &= \sum_{y=S}^{\infty} (y - S) \cdot p_y^{LT+R} \\ &= \mu_{LT+R} - S - \sum_{y=0}^{S} (y - S) \cdot p_y^{LT+R} \\ &= \mu_{LT+R} - S + E[S - Y_{LT+R}]^+ \\ &= 411.6 - 439 + 28.264 \\ &= 0.864 \end{aligned}$$

This yields expected cost per review period of

$$Z(S) = 1 \cdot 28.264 + 10 \cdot 0.864 = 36.904$$

The expected profit per review period is

$$\begin{aligned}\Pi(S) &= (r-c)\cdot\mu_R - Z(S)\\ &= (80-69)\cdot 68.6 - 36.90\\ &= 717.70\end{aligned}$$

ii. The optimal order-up-to-level $S^\star$ for normally distributed demand is

$$S^\star = F^{-1}\left(\frac{p}{p+h}\right) = F^{-1}(0.909) = 9.8\cdot(35+7) + 1.34\cdot\sqrt{(35+7)\cdot 11.56} = 441.$$

The expected cost per review period are given by:

$$\begin{aligned}Z(S^\star) &= (h+p)\cdot f_{N(0,1)}(z)\cdot\sigma_{LT+R}\\ &= (1+10)\cdot f_{N(0,1)}(0.909)\cdot\sqrt{42\cdot 11.56}\\ &= 11\cdot 0.1626\cdot 22\\ &= 39.35.\end{aligned}$$

The expected profit per review period is

$$\begin{aligned}\Pi(S) &= (r-c)\cdot\mu_R - Z(S)\\ &= (80-69)\cdot 68.6 - 39.35\\ &= 715.25.\end{aligned}$$

In this case the results for both types of distribution are very close. The Normal distribution is much easier to apply, than the Poisson distribution, where the pdf has to be solved explicitly numerous times. In general, the Normal distribution can be applied if

1. The average demand over LT+R is not to small. (E.g. > 30)
2. The probability for negative demand, $P(Y_{LT+R} < 0)$, is sufficiently small. I.e. the vartiance-to-mean ratio is small.
3. The histogramm of historic demand resembles the shape of a Normal distribution.

Solution to Exercise 3:

i. An order x_{16} that is placed in period 16 arrives at the end of period 24. Therefore, it does not affect the inventory level (and thus the inventory holding or backorder penalty cost) of any periods prior to period 24. Next, we will argue, why x_{16} has now effect on the cost in any periods later than period 24. In the next review period, i.e. period 17, we issue a new order x_{17}, which is going to arrive at the end of period 25. Since we follow an order-up-to policy, we issue an order that is equal to the difference between S and the inventory position in period 17, i.e. $x_{17} = S - IP_{17}$. However, recall that $IP_{17} = IP_{16} + x_{16} - y_{16}$. Now observe, that if we choose a large order, say x'_{16}, the inventory position in period 17 will be higher and the difference between S and IP_{17} will be small. If we choose a smaller order, say x''_{16}, then inventory position in period 17 will be correspondingly smaller by the difference between $x'_{16} - x''_{16}$ and the difference between S and IP_{17}, and thus x_{17}, will be larger by the same amount. So to say, the order in period 17 compensates for any decision in period 16. Since x_{17} arrives at the end of period 25, any decision in period 16 has no effect on the cost in periods 25 and later. The only period whose cost are affected by x_{16} is period 24.

ii. The optimal order-up-to-level $S^\star$ in a periodic review inventory system is the same for all periods. We use the standart approach, i.e.

$$S^\star = F^{-1}\left(\frac{p}{p+h}\right) = F^{-1}\left(\frac{2}{2+0.5}\right) = F^{-1}(0.8) = 10 \cdot (8+1) + 0.84 \cdot \sqrt{8+1} \cdot 4 = 100.08.$$

iii. The order in period 16 affects the inventory level and thus the cost in period 24. It holds that

$$I_{24} = IP_{16} + x_{16} - Y^{8+1}. \tag{9}$$

If we could order as usual, we would set $IP_{16} + x_{16} = S$ and compute the expected overage and underage at the end of period 24 just as $E\left[S - Y^9\right]^+$ and $E\left[Y^9 - S\right]^+$. But, in this situation we cannot order $x_{16} = S - IP_{16}$. Therefore, we first note that

$$IP_{16} = IP_{15} + x_{15} - y_{15} \tag{10}$$

$$= S - y_{15}, \tag{11}$$

since we can order optimally in period 15.

Insertion of (11) in (9) yields

$$I_{24} = S - y_{15} + x_{16} - Y^9 \tag{12}$$

$$= S + x_{16} - Y^{10}. \tag{13}$$

The expected underage quantity at the end of period 24 is

$$E\left[I_{24}\right]^- = E\left[Y^{10} - S - x_{16}\right]^+ \tag{14}$$

$$= \int_{y=S+x_{16}}^{\infty} \left(y - (S + x_{16})\right) f_{10}(y)\, dy \tag{15}$$

$$= \sigma_{10} \cdot L\left(\frac{S + x_{16} - \mu_{10}}{\sigma_{10}}\right) \tag{16}$$

For $x_{16} = 10$:

$$E\left[I_{24}\right]^- = \sqrt{10} \cdot 4 \cdot L\left(\frac{100.2 + 10 - 10 \cdot 10}{\sqrt{10} \cdot 4}\right) \tag{17}$$
$$= 12.65 \cdot L\left(0.81\right) \tag{18}$$
$$= 12.65 \cdot 0.1181 \tag{19}$$
$$= 1.49 \tag{20}$$

For $x_{16} = 100$:

$$E\left[I_{24}\right]^- = \sqrt{10} \cdot 4 \cdot L\left(\frac{100.2 + 100 - 10 \cdot 10}{\sqrt{10} \cdot 4}\right) \tag{21}$$
$$= 12.65 \cdot L\left(7.921\right) \tag{22}$$
$$\approx 12.65 \cdot 0.00 \tag{23}$$
$$= 0.00 \tag{24}$$

The expected overage quantity at the end of period 24 is

$$E\left[I_{24}\right]^+ = E\left[S + x_{16} - Y^{10}\right]^+ \tag{25}$$
$$= \int_{y=-\infty}^{S+x_{16}} \big(S + x_{16} - y\big) f_{10}\left(y\right) dy \tag{26}$$
$$= \int_{y=-\infty}^{\infty} \big(S + x_{16} - y\big) f_{10}\left(y\right) dy - \int_{y=S+x_{16}}^{\infty} \big(S + x_{16} - y\big) f_{10}\left(y\right) dy \tag{27}$$
$$= \big(S + x_{16}\big) \int_{y=-\infty}^{\infty} f_{10}\left(y\right) dy - \int_{y=-\infty}^{\infty} y \cdot f_{10}\left(y\right) dy \tag{28}$$
$$+ \int_{y=S+x_{16}}^{\infty} \big(y - \left(S + x_{16}\right)\big) f_{10}\left(y\right) dy \tag{29}$$
$$= S + x_{16} - \mu_{10} + \sigma_{10} \cdot L\left(\frac{S + x_{16} - \mu_{10}}{\sigma_{10}}\right). \tag{30}$$

For $x_{16} = 10$:

$$E\left[I_{24}\right]^+ = 100.2 + 10 - 100 + 1.49 \tag{31}$$
$$= 11.69 \tag{32}$$

For $x_{16} = 100$:

$$E\left[I_{24}\right]^+ = 100.2 + 100 - 100 + 0 \tag{33}$$
$$= 100.2 \tag{34}$$

The expected cost for $x_{16} = 10$ are:

$$Z(x_{16} = 10) = h \cdot E\left[S + x_{16} - Y^{10}\right]^{+} + p \cdot E\left[Y^{10} - S - x_{16}\right]^{+} \tag{35}$$
$$= 0.5 \cdot 11.69 + 2 \cdot 1.49 \tag{36}$$
$$= 8.825 \tag{37}$$

The expected cost for $x_{16} = 100$ are:

$$Z(x_{16} = 10) = 0.5 \cdot 100.2 + 2 \cdot 0 \tag{38}$$
$$= 50.1 \tag{39}$$

Therefore, we should chose to order 10 units in period 16.

iv. On average, the order quantity in a period is $\mu = 10$. Under an optimal $S^{\star} = 100.2$ the expected cost is

$$Z(S^{\star}) = (h + p) \cdot f_{N(0,1)}(z) \cdot \sigma_{LT+1} \tag{40}$$
$$= (2 + 0.5) \cdot f_{N(0,1)}(0.84) \cdot \sqrt{9} \cdot 4 \tag{41}$$
$$= 2.5 \cdot 0.280 \cdot 12 \tag{42}$$
$$= 8.40. \tag{43}$$

Ordering 10 units in period 16 results in expected extra cost in perod 24 of $8.825 - 8.40 = 0.425$.